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A fuzzy approach to the convective longitudinal fin array design

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Abstract

This work considers an optimum design problem for the longitudinal fin array with constant heat transfer coefficient in a fuzzy environment, where rigid requirements to strictly satisfy the total fin volume and array width and maximize the heat dissipation rate are softened. The proposed method shows that the fuzzy fin array design problem can be converted into a regular min–max type optimization problem by employing the tolerance approach. An entropic regularization technique is then applied to solve the resulting optimization problem. Some computational results are presented to illustrate the theory and solution procedure. A comparison of results from the regular non-fuzzy optimal model and the fuzzy model is also included.

Keywords: Fin array; Fuzzy approach; min–max optimization; Tolerance approach; Entropic regularization technique

1. Introduction

It is well known that fins have been widely used in heat transfer devices to increase the heat dissipation rate. The fin design optimization problem has great interest and much work has been done to improve fin performance in the literature. There are two different approaches for these fin optimization problems. In the first approach, the minimum volume or least fin fabrication material is expanded for the prescribed heat dissipation. The other approach considers finding the maximum heat dissipation for a given fin volume. Most of these works were performed using a single fin or spine. A thorough treatment for the optimum single fin or spine design was given by Kern and Kraus [1]. However, fin arrays are used more often than single fins or spines in practical engineering applications. Dhar and Arora [2] presented methods for carrying out the minimum fin array weight design with triangular cross-section for a flat surface, cylindrical surface, and rectangular cross-section over a flat surface. Bar-Cohen [3] studied the fin thickness for a natural convective fin array with a rectangular profile and concluded that in natural convection arrays, superior thermal performance is generally associated with relatively thick fins. A least material optimization investigation of a vertical plate-fin heat sink in natural convective heat transfer was given by Iyengar and Bar-Cohen [4]. The optimum dimensions of the fin space and fin thickness were studied using the Nusselt number correction by Bar-Cohen and Rohsenow. Recently, Bar-Cohen et al. [5] extended the least-material single rectangular plate-fin analysis to multiple fin arrays, using a composite Nusselt number correlation to find the globally best thermal design for the natural convective heat sinks. In their work, optimum fin array design was combined with the least fin material and optimal spacing.

The development of fuzzy set theory has forged a new way to deal with imprecision and vagueness in information since 1965 [6]. It has applications in many different practical fields, such as electrical, air-conditioning and refrigeration, aerospace, chemical, transportation, power industries. The first attempt to apply the fuzzy set theory to a heat transfer problem was Zhang and Chung [7]. The optimum fin height and fin thickness were investigated for a given heat dissipation rate on a single rectangular longitudinal convective fin. The same analysis was applied to a conical convective spine by Chen and Lin [8]. Latter, Chung and Chen [9,10] applied

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Nomenclature

b	half fin thickness	m
$\widetilde{D}$	fuzzy decision	
d	half space between fins	m
H	height of the fin array	m
h	average heat transfer coefficient	$W \cdot m^{-2} \cdot K^{-1}$
h_f	heat transfer coefficient of fin surface	$W \cdot m^{-2} \cdot K^{-1}$
h_H	heat transfer coefficient of fin base	$W \cdot m^{-2} \cdot K^{-1}$
k	thermal conductivity of the fin	$W \cdot m^{-1} \cdot K^{-1}$
L	length of the fin array	m
N	total number of fins	
Q	heat transfer rate of the fin array	W
T	temperature of fin surface	K
t	tolerance	
V	total volume of the fin array	m^3
W	width of fin array	m

Greek symbols

θ	temperature difference between fin surface and ambient	K
μ	membership function	

Subscripts, superscript

H	fin base
∞	ambient
$*$	prescribed value

the fuzzy optimization technique to the rectangular longitudinal fin and cylindrical spine designs with different models. The first paper that studied the fuzzy fin array optimization was presented by Chung et al. [11]. In their work, a four-fin radiating fin array system was developed and optimized by employing the fuzzy approach, in which the total weight and horizontal dimension of the fin system were minimized. This study considers fuzzy optimal longitudinal fin array design for a given array volume and a prescribed array width in a natural convective environment where the heat transfer coefficient are averaged and assumed constant.

As mentioned above, this work studies fuzzy optimum natural convection fin array design with constant heat transfer coefficient. We aim to maximize the heat transfer rate for the given fin volume and array width in accordance with a prescribed tolerance. In our problem, the total volume V and the width W of the fin array are considered to be "approximately" equal to prescribed values V^* and W^*. To investigate the solution method for solving the fuzzy convective fin array design problem, mathematical model thermal analysis is presented in Section 2. Applying the fuzzy set theory, in Section 3 we show that the fuzzy convective fin array design problem can be converted into a regular min–max nonlinear programming problem. An entropic regularization technique [12,13] is then applied to solve the resulting optimization problem. Numerical results and discussion are provided in Section 4 to confirm the efficiency of the proposed method. Section 5 concludes this paper by making some remarks.

2. The thermal analysis

Consider the longitudinal fin array with rectangular cross-section as shown in Fig. 1. The heat dissipation of the fin array is comprised of two components. One is the heat transfer from the surface between the fins, and the other is heat

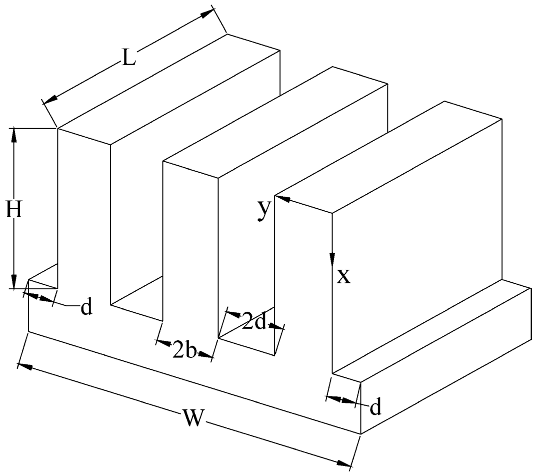

Fig. 1. Schematic of an array with longitudinal rectangular fins.

dissipation from the surface of each fin in the array. In this case, the heat dissipation rate can be described as follows [1]

$$Q = N\left[2dh_H(T_H - T_\infty) + \int_0^H 2h_f(T - T_\infty)\,\mathrm{d}x\right] \tag{1}$$

where N is the total number of fins, d is the half space between the fins, H is the fin height, h_H and h_f are the convection coefficient at the fin base and surface, respectively, T_H is the temperature at the fin base, and T_∞ is the surrounding temperature. In this study, the functional relations of the width, W, and the total volume, V, of the fin array are given as follows for unit fin length $L = 1$.

$$W = N(2b + 2d) = 2N(b + d) \tag{2}$$

$$V = N(2bHL) = \frac{bHW}{b + d} \tag{3}$$

where b is the half fin thickness.

Assuming that the heat conduction along a single rectangular fin is one-dimensional, i.e., along the x direction. The heat entering the fin by conduction is equal to the heat dissipated by convection to the surrounding air at temperature T_∞. Applying the Gardner assumptions, the energy equation for a convective fin can be written as

$$b\frac{d^2T}{dx^2} = \frac{h_f}{k}(T - T_\infty) \tag{4}$$

where k is thermal conductivity. Let $\theta = T - T_\infty$, the temperature profile for a single rectangular fin for above equations is obtained as follows.

$$\theta(x) = \theta_H \frac{\cosh(\sqrt{h_f x^2/kb})}{\cosh(\sqrt{h_f H^2/kb})} \tag{5}$$

Moreover, by assuming an average heat transfer coefficient $h = h_f = h_H$, the overall heat flowing through the base of the fin array, Eq. (1), can be calculated as

$$Q = \frac{hW\theta_H}{b+d}\left[d + \sqrt{\frac{kb}{h}}\tanh\left(\sqrt{\frac{hH^2}{kb}}\right)\right] \tag{6}$$

Therefore, the optimal design problem of finding the maximum heat dissipation for a given volume of the rectangular fin array can be considered as follows:

maximize

$$Q = \frac{hW\theta_H}{b+d}\left[d + \sqrt{\frac{kb}{h}}\tanh\left(\sqrt{\frac{hH^2}{kb}}\right)\right] \tag{7}$$

subject to

$$V = \frac{bHW}{b+d} = V^* \tag{8}$$

$$W = 2N(b+d) = W^* \tag{9}$$

$$b, d, H, N > 0 \tag{10}$$

where Eq. (7) is the objective function and Eqs. (8)–(10) are the rigid constrains. Both V^* and W^* are prescribed values.

A solution (b, d, N) of the problem, Eqs. (7)–(10), will be obtained to produce the maximum heat dissipation Q for the given total fin volume V^*, array width W^*, and fin height H. On the other hand, given the total fin volume V^*, array width W^*, and half fin thickness b, an optimal solution (H, d, N) will be obtained to provide the maximum heat dissipation Q. It should be noted that for the solution (b, d, N) or (H, d, N) the value of N is a real number instead of an integer. Since it is physically meaningless if N is not an integer for this problem, the constraint of N being an integer should be added. In this case, the value of the maximum heat dissipation Q becomes smaller due to the added constraint and its correlation with the other rigid constrains, that is Eqs. (8) and (9). In this work we consider finding better solutions for the heat dissipation Q by softening the constraints of Eqs. (8) and (9) under the condition of N being an integer. To be more precise, the total volume V and the width W of the array are considered to be "approximately equal to" the prescribed values V^* and W^*. A corresponding fuzzy mathematical programming problem is discussed in the next section.

3. The fuzzy mathematical program model and solution procedure

As mentioned above, in this work the rigid requirements that strictly satisfy the total fin volume and array width to maximize the heat dissipation rate for Eqs. (7)–(10) are to be softened. In this case, the fin array optimum design problem we considered can be modeled as a fuzzy mathematical program and described as follows:

maximize

$$Q = \frac{hW\theta_H}{b+d}\left[d + \sqrt{\frac{kb}{h}}\tanh\left(\sqrt{\frac{hH^2}{kb}}\right)\right] \tag{11}$$

subject to

$$V = \frac{bWH}{b+d} \cong V^* \tag{12}$$

$$W = 2N(b+d) \cong W^* \tag{13}$$

$$b, d, H, N > 0 \tag{14}$$

where Eqs. (12) and (13), are fuzzy equalities and "≅" denotes the fuzzified version of "=" with the linguistic interpretation "approximately equal to". Eq. (14) is a regular inequality. In our problem, we aim to maximize the heat transfer rate and the constraints are softened such that the total fin volume and array width are approximately equal to the prescribed volume, V^*, and width, W^*.

The fuzzy equality, Eq. (12), actually determines a fuzzy set, whose membership function is denoted by μ_V. The membership grade $\mu_V(b, d, W, H)$ can be interpreted as the degree to which the regular equality $V = V^*$ is satisfied. To specify the membership function μ_V, it is commonly assumed that $\mu_V(b, d, W, H)$ should be zero if the regular equality $V = V^*$ is strongly violated, and one if it is satisfied. This leads to a membership function in the following form

$$\mu_V(b, d, W, H) = \begin{cases} 0, & \text{if } V > V^* + t_v, \text{ or } V < V^* - t_v \\ \frac{1}{t_v}|V - V^*|, & \text{if } V^* - t_v \leqslant V \leqslant V^* + t_v \\ 1, & \text{if } V = V^* \end{cases} \tag{15}$$

where $t_v \geqslant 0$ is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy equality $V = V^*$.

Similarly, the other fuzzy equality constraint, Eq. (13), can be represented as a fuzzy set with corresponding membership function $\mu_W(b, d, N)$ and can be described as follows:

$$\mu_W(b, d, N) = \begin{cases} 0, & \text{if } W > W^* + t_w \\ & \text{or } W < W^* - t_w \\ \frac{1}{t_w}|W - W^*|, & \text{if } W^* - t_w \leqslant W \leqslant W^* + t_w \\ 1, & \text{if } W = W^* \end{cases} \tag{16}$$

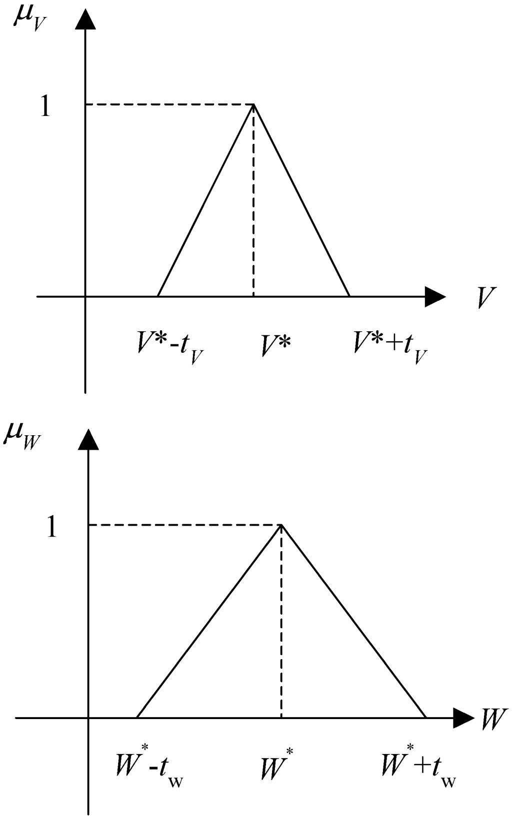

Fig. 2. Membership function for μ_V an μ_W.

where $t_w \geqslant 0$ is the tolerance level (determined by the decision maker). Fig. 2 shows the membership functions μ_V and μ_W.

Based on the concept that fuzzy constraints should yield a fuzzy objective [14], two possible extreme points Q_0 and Q_1 are provided for constructing the membership function of the objective by solving

$Q_0 =$ maximize

$$Q = \frac{hW\theta_H}{b+d}\left[d + \sqrt{\frac{kb}{h}}\tanh\left(\sqrt{\frac{hH^2}{kb}}\right)\right] \quad (17)$$

subject to

$$V = \frac{bHW}{b+d} = V^* \quad (18)$$

$$W = 2N(b+d) = W^* \quad (19)$$

$$b, d, H, N > 0 \quad (20)$$

and

$Q_1 =$ maximize

$$Q = \frac{hW\theta_H}{b+d}\left[d + \sqrt{\frac{kb}{h}}\tanh\left(\sqrt{\frac{hH^2}{kb}}\right)\right] \quad (21)$$

subject to

$$V^* - t_V \leqslant \frac{bWH}{b+d} \leqslant V^* + t_v \quad (22)$$

$$W^* - t_w \leqslant 2N(b+d) \leqslant W^* + t_w \quad (23)$$

$$b, d, H, N > 0 \quad (24)$$

The membership function of the fuzzy objective is defined as follows:

$$\mu_Q(b,d,W,H) = \begin{cases} 1, & \text{if } Q > Q_1 \\ 1 - \frac{Q_1 - Q}{Q_1 - Q_0}, & \text{if } Q_0 \leqslant Q \leqslant Q_1 \\ 0, & \text{if } Q < Q_0 \end{cases} \quad (25)$$

Let a fuzzy decision $\widetilde{D}$ of the above problem be defined as the fuzzy set resulting from the intersection of the fuzzy objective and the fuzzy equalities with a corresponding membership function

$$\mu_{\widetilde{D}}(b,d,W,H,N) = \min\{\mu_Q(b,d,W,H), \mu_V(b,d,W,H), \mu_W(b,d,N)\} \quad (26)$$

According to the fuzzy set theory [15,16], a solution, say $(b_{\text{op}}, d_{\text{op}}, W_{\text{op}}, H_{\text{op}}, N_{\text{op}})$, of the convective optimum fin design problem, Eqs. (11)–(14), can be taken as the solution with the highest membership in the fuzzy decision set $\widetilde{D}$ and obtained by solving the following nonlinear programming problem.

$$\max\min\{\mu_Q(b,d,W,H), \mu_v(b,d,W,H), \mu_W(b,d,N)\} \quad (27)$$

which is equivalent to the following "min–max" problem

$$-\min\mu'_D(b,d,W,H,N) = \max\{-\mu_Q(b,d,W,H), -\mu_V(b,d,W,H), -\mu_w(b,d,N)\} \quad (28)$$

From the above procedure, we see that the convective optimum fin array design problem, Eqs. (7)–(10), can eventually be reduced to a nonlinear programming problem, Eq. (28). One major difficulty encountered in developing solution methods for solving the "min–max" problem, Eq. (28), is the non-differentiability of the max function $\mu'_D(b,d,W,H,N)$. A distinct feature of the recent development centers on the idea of developing "smooth algorithms" [17]. Among them, a class called "regularization methods" has been developed based on approximating the max function $\mu'_D(b,d,W,H,N)$ by certain smooth function [18]. Here we adopt the newly proposed "entropic regularization procedure" [12,13]. This procedure guarantees that for an arbitrarily small $\varepsilon > 0$, an ε-optimal solution of the min–max problem, Eq. (28) can be obtained by solving the following problem

$$-\min\mu_r(b,d,W,H,N) = \frac{1}{r}\ln\{\exp[r(-\mu_Q(b,d,W,H))] + \exp[r(-\mu_V(b,d,W,H))] + \exp[r(-\mu_w(b,d,N))]\} \quad (29)$$

with a sufficiently large r, where $\mu_r(b, d, W, H, N)$ is a smooth function which approximates $\mu'_D(b, d, W, H, N)$ uniformly and accurately when r is taken sufficiently large. It should be noted that in practice a sufficiently accurate approximation could be obtained using a moderately large r. Also because $\mu_r(b, d, W, H, N)$ appears in a special "log-exponential" form, it is highly smooth and avoids most overflow problems in computation. The BFGS subroutine is then adopted here for solving the optimization problem, Eq. (29). This subroutine is a modified Newton's method, which replaces the Jacobian matrix in Newton's method with an approximation matrix that is updated at each iteration.

4. Results and discussion

This work examined the fuzzy optimum design of a natural convective fin array with constant heat transfer coefficient. In our problem the total volume V and the width W of the fin array are considered to be "approximately" equal to the prescribed values V^* and W^*. We aim to maximize the heat transfer rate for the given fin volume and array width which are allowed to have certain tolerance. To provide some idea on the physical significance of various variables, we considered the case of a copper fin array convecting into air. Properties used for the numerical implementation include the thermal conductivity, k, being equal to 382 $W{\cdot}m^{-1}{\cdot}K^{-1}$, the average heat transfer coefficient, h, being 3 $W{\cdot}m^{-2}{\cdot}K^{-1}$, the temperature difference θ_H which is 100 °C, the prescribed array width W^* being 0.06 m, the tolerance for the array width t_w and the array volume t_V being 1% of W^* and 1% of V^*, respectively.

The comparison of the maximum heat dissipation rate Q, the number of fins N, and the half space between fins d for the regular non-fuzzy optimal model and the fuzzy model are plotted in Figs. 3–5. The prescribed array volume V^* is equal to 0.0015 m^3 in these cases. The aspect ratio of the fin, $H/2b$, is defined by the fin height divided by the fin thickness. For the regular non-fuzzy model, the number of fin N was treated as integer numbers and both the prescribed array volume V^* and array width W^* are rigid constraints during the process of optimal procedure. Because of the soft constraint of W^* and V^* in the fuzzy model, the optimal number of the fin, N, is one or two more than that of the regular optimization model in this study as shown in Fig. 3. The half space between fins, d, was also affected by the change in the numbers of fins N from the regular into the fuzzy model, as displayed in Fig. 4. The half space between fins, d, of the regular model was larger than that of the fuzzy model because fewer fin numbers in the same space for the regular model. The increasing numbers of fins and decreasing half space between the fins combines the prescribed array volume soft constraint effects to provides a better maximum heat dissipation rate, Q, for the fuzzy model, as illustrated in Fig. 5.

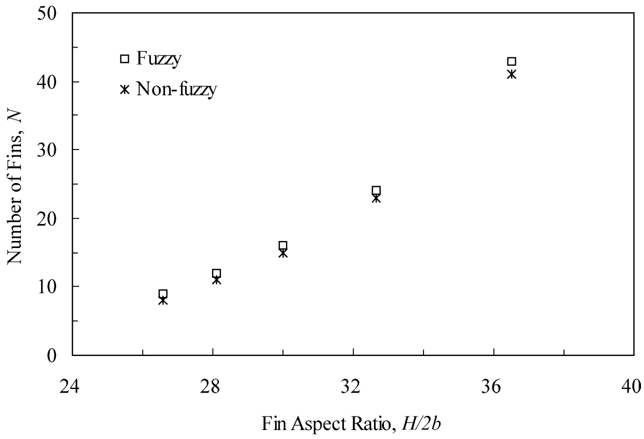

Fig. 3. Number of fins for the regular non-fuzzy model and the fuzzy model.

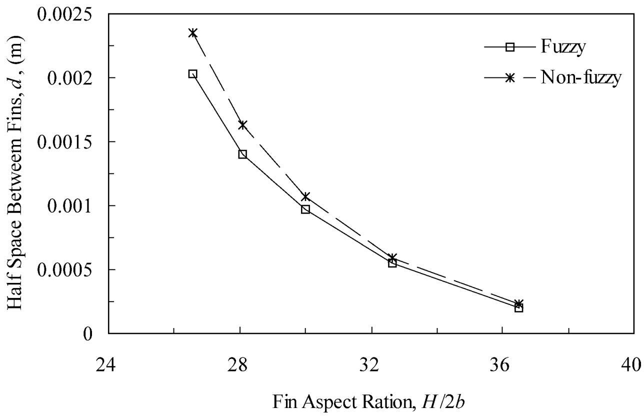

Fig. 4. The half fin thickness for the regular non-fuzzy model and the fuzzy model.

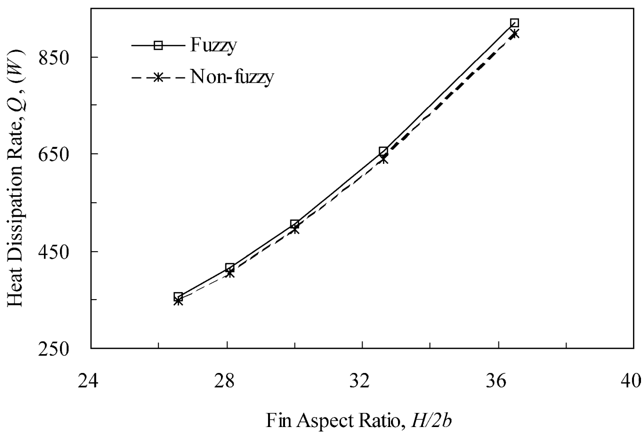

Fig. 5. Optimal heat dissipation rate for the regular non-fuzzy model and the fuzzy model.

Figs. 6–8 show the heat dissipation rate, number of fins, and half space between fins results as functions of the aspect ratios for the fuzzy optimization model. The given array volumes V^* are equal to 0.0007, 0.001, and 0.0015 m^3, respectively. The optimal heat dissipation rate, q, as well as the number of fin, N, increase as the aspect ratio increases for a given array width and a given array volume, as dis-

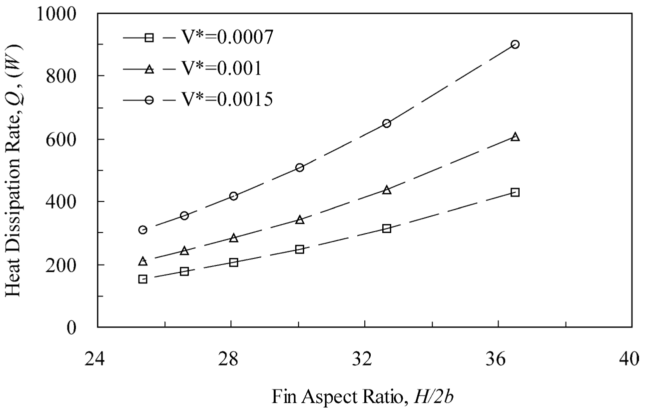

Fig. 6. Optimal heat dissipation rate as function of fin aspect ratio for the fuzzy model.

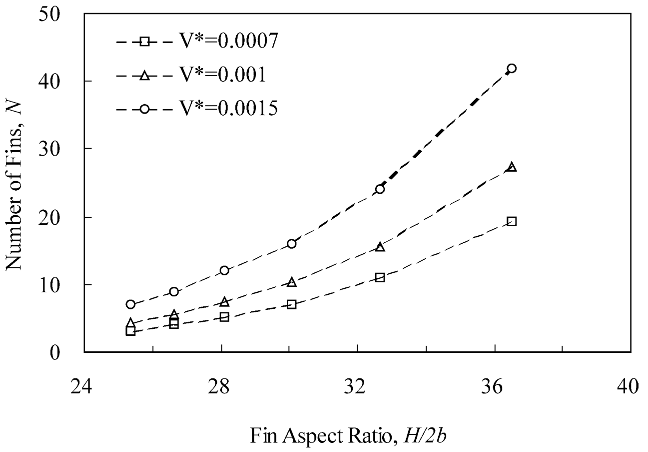

Fig. 7. Number of fins as function of fin aspect ratio for the fuzzy model.

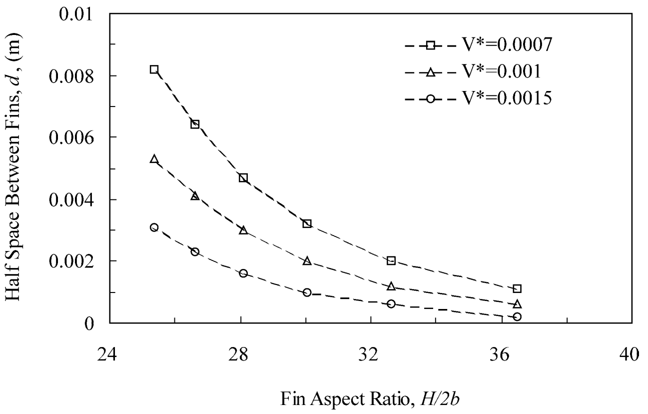

Fig. 8. The half space between fins as function of fin aspect ratio for the fuzzy model.

played in Figs. 6 and 7. In the optimization process, two conditions arise when the aspect ratio, $H/2b$, increases with the constraints of a given array width and a given array volume. In our cases, one is the number of fins, N, increases which causes the half space between fins, d, to decrease. The other is the half space between the fins, d, increasing and the number of fins, N, decreasing. Because of the av-

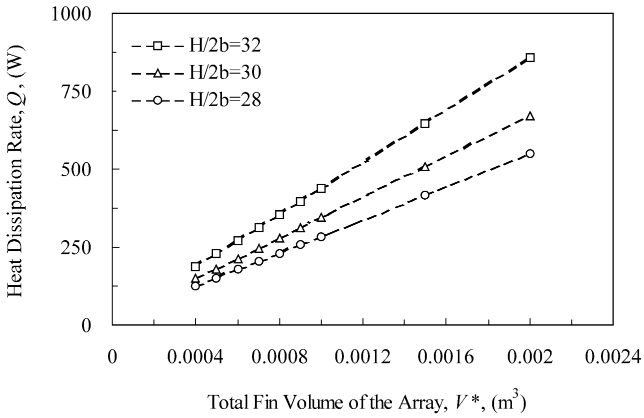

Fig. 9. Optimal heat dissipation rate as function of the array volume for the fuzzy model.

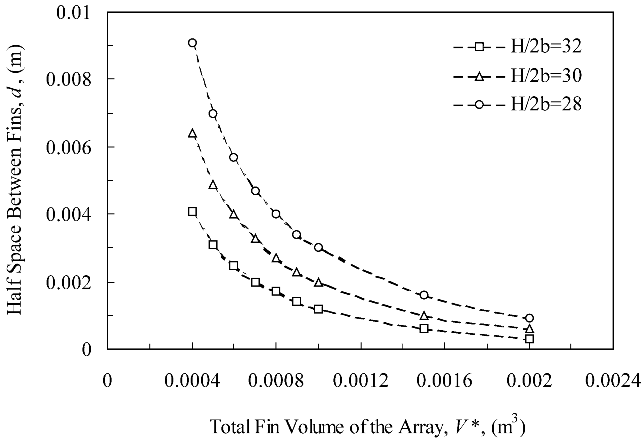

Fig. 10. Half space between fins as function of the array volume for the fuzzy model.

erage heat transfer coefficient being constant assumption, $h = h_f = h_H$, the optimal heat dissipation rate is obtained for the case with the numbers of fins increasing and the half space between the fins decreasing, the heat dissipation rate is greater from the increasing fin surface area than that from the fin base. As anticipated, a different trend that the half space between fins d decreases with the aspect ratio increasing is presented in Fig. 8.

Figs. 9 and 10 show the heat dissipation rate and half space between fins results for the fuzzy optimization models with different aspect ratios. The number of fins increases almost linearly as the array volume increases when the aspect ratio and the array width are fixed. Therefore, the optimal heat dissipation increases almost linearly with the array volume increases, as shown in Fig. 9. To optimize the heat dissipation rate, the half space between the fins must be in inverse proportion to the number of fins at a constant aspect ratio and fixed array width. Hence, the half space between fins decreases with increasing array volume as a result of increasing the number of fins, N, and decreasing the half space between the fins, d, as illustrated in Fig. 10. This suggests that the optimal dimensions for the fin array design problem should have larger aspect ratio, $H/2b$, and number of fins, N, with a smaller half space between the fins, d, to provide a better heat dissipation rate for the same array volume and array width.

All computations in this study were carried out at RISC 6000 workstations. The cpu times required for the calculations were between 10 to 16 seconds for each case.

5. Conclusion

In this work a convective fin array design problem in a fuzzy environment was studied. The volume and the width of the fin array were allowed to have a certain tolerance for finding the maximum heat transfer rate with average heat transfer coefficient. Using the tolerance approach showed that the fuzzy convective fin array design problem can be converted into a regular min–max programming problem. An entropic regularization technique was then applied to solve the resulting optimization problem. Because of the entropic regularization technique, only a commonly used BFGS subroutine was required in our implementation.

A comparison of the thermal performance, number of fins, and half space between the fins were given to the optimum result from the fuzzy optimization and non-fuzzy optimization. The soft array volume and array width constraints allowed one or two more fins to be included in the array to dissipate more heat. At a constant averaging heat transfer coefficient, the heat dissipation rate and number of fins increases with increasing fin aspect ratio when the array width is limited. The half space between fins decreases with increasing array volume for the case with constant aspect ratio.

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